A Bond Graph Approach to the Modeling of General Multibody Dynamic Systems

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A vector bond approach which effectively leads to a compact form of Hamiltonian bond graph structure and naturally to Hamilton's equation of motion is proposed for the modeling of general multibody dynamic system. The methods for determining required bond graph elements are formulated in terms of kinematic influence coefficients. All moduli of transformers and their time rate of changes are found by pure vector (matrix) operations for the readiness for computer simulation of the resulting bond graph.

Key Words: Bond-Graph, KIC (Kinematic Influence Coefficient)

1. Introduction

The visualization of the energetic structure of a dynamic system is very useful in various aspects of system engineering, e. g., system analysis, synthesis, and control. Since Paynter (1960) first proposed the bond graph as a unified method of modeling physical systems in early sixties, bond graph approach has been known to be a convenient tool for the visualization of the energetic structure of a dynamic system.

Its applications in various engineering disciplines have been exponentially increased as being reflected in the literature reviewed by Bos and Breedveld (1985). But the conventional bond graph method, even with the power of visualizing energetic structure, does not provide a straightforward way of dynamic modeling of a general multibody dynamic system. This is mainly because of the fact that as the number of bodies in a system increases, the detailed dynamic interaction structure of the system becomes quite complicated due to natural existence of many dependent inertial elements resulting from kinematic constraints involved in the system.

Some of the key efforts made to resolve this difficulty are as follows: Karnopp (1969) introduced an important concept of power conserving transformation into bond graph and helped Rosenberg (1972) to establish a foundation toward multiport mechanics. Utilizing generalized coordinate, Brown (1972) proposed Lagrangian bond graph which is directly compatible with Lagrange's equation. Given a usual bond graph, Rosenberg (1977) devised, quite differently from Brown (1972), a new procedure of causality assignment which facilitated finding state equation by using Lagrange's equation. Later, Rosenberg (1978) asserted the good aspect of the bond graph method in modeling multibody dynamic systems, recommended the usage of proper stiffness elements as a way of getting around the difficulties caused by dependent inertial elements, and incorporated the idea into Rosenberg and Margolis (1979). Allen and Dubowsky (1977) were practically the first who attempted to model a general multibody system using bond graph. Allen (1979) formulated Lagrange's equation from bond graph, and Allen (1981) also reformulated the dynamics of mechanisms in accelerating coordinate frame by introducing comprehensive notation of velocities. By following and expanding Allen's works, Bos and Tiernego (1985) and Tiernego and Bos (1985) developed a systematic algorithm for the

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modeling and simulation of a general interconnected multibody dynamic system.

This paper also attempts to formulate kinematics and dynamics of a general multibody dynamic system by using bond graph. The approach developed here may be considered to be dual to the one used in Bos and Tiernego (1985) and Tiernego and Bos (1985) in a sense that in this paper, rather than using accelerating frame, all the kinematic formulations are performed in inertial frame by using the concept of kinematic influence coefficient(KIC), which was first introduced in dynamic system modeling by Paul (1975) and Benedict and Tesar (1978a, 1978b). Extension of the concepts was made in Thomas (1982), Cho et al. (1989), Cho (1994), and Paul (1989). Similarly to the concept of the first order KIC, Kane and Levinson (1985) defined the concept of the linear and angular partial velocities. In bond graph terminology, the first order KIC is equivalent to the modulus of a transformer.

This paper is organized as follows. In the following section, the kinematics of a general constrained multibody system is briefly formulated in terms of KIC. Full details can be found in Cho (1994). In Sec. 3, the hamiltonian bond graph is developed, and finally concluding comment comes in Sec. 4.

2. Kinematics in Terms of KIC

2.1 Kinematic constraints

Consider a interconnected multibody dynamic system which contains several closed kinematic chains as shown in Fig. 1 Note in Fig. 1 that all the joints are assumed to be simple[1] without loss of generality and numbered by using positive integers. The set of joint indices used to index all the simple joints in the system will be denoted by J, i. e., $J=\{j_i : j_i \in Z^+ \text{ with } i=1, \cdots, J\}$ where

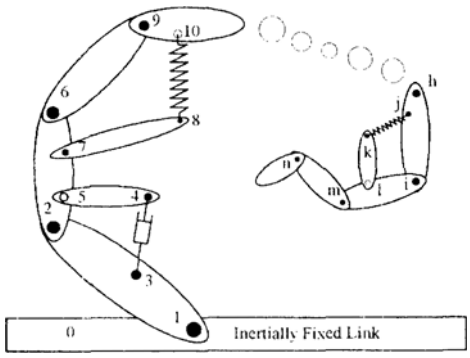

Fig. 1 Conceptual interconnected multibody system.

Z^+ denoote the set of positive integers and J is the total number of simple joints.

A set of independent holonomic constraints in terms of proper Lagrangian coordinates $\Psi \in R^J$ may be expressed in vector form[2] as

$$\boldsymbol{f}(\boldsymbol{\Psi}) = \boldsymbol{0} \tag{1}$$

where $f_i (i=1, \cdots, M)$ are at least twice differentiable functions w. r. t. (with respect to) Ψ. Assuming the system possesses N degrees of freedom, Eq. (1) may be rewritten as

$$\boldsymbol{f}(\boldsymbol{\phi}, \ \boldsymbol{q}) = \boldsymbol{0} \tag{2}$$

where $\boldsymbol{\phi} \in R^N$ denotes the (independent) generalized coordinate vector selected from $\boldsymbol{\Psi}$ and the vector $\boldsymbol{q} \in R^M$ represents any dependent redundant set of coordinates involved in $\boldsymbol{\Psi} \in R^J$ such that $J = N + M$. The selections of the vectors $\boldsymbol{\phi}$ and $\boldsymbol{q}$ from Lagrangian coordinate vector $\boldsymbol{\Psi}$ may be formally described as $\phi_i = \Psi_{p(i)}$ with $i=1, \cdots, N$ and $q_i = \Psi_{s(i)}$ with $i=1, \cdots, M$ where two index functions, $p(i)$ and $s(i)$ are defined as $p(\cdot) : \{1, \cdots, N\} \rightarrow J$ and $s(\cdot) : \{1, \cdots, M\} \rightarrow J$ and their inverse relationships $\underline{p}(\cdot)$ and $\underline{s}(\cdot)$ are also defined as $\underline{p}(\cdot) : J \rightarrow \{1, \cdots, N\}$ and $\underline{s}(\cdot) : J \rightarrow \{1, \cdots, C\}$.

Differentiating Eq. (2) w. r. t. time and solving for $\dot{\boldsymbol{q}}$ lead to

$$\dot{\boldsymbol{q}} = [\tilde{\boldsymbol{G}}] \dot{\boldsymbol{\phi}} \tag{3}$$

where $[\tilde{\boldsymbol{G}}] \in R^{M \times N}$ will be called the first order internal kinematic influence coefficient(IKIC) matrix and is defined as

[1] A joint is called simple if it allows only one degree of freedom of relative motion. Any compound joint like ball and socket joint can be modeled by series of simple joints.

[2] In this paper, a bold face letter denotes a vector expressed in column-wise form.

$$[\tilde{G}]=\left[\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\phi}}\right] \tag{4}$$

$$=-\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right]^{-1}\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}\right] \tag{5}$$

where $[\partial \boldsymbol{f}/\partial \boldsymbol{q}]$ is assumed to be nonsingular without loss of generality because proper selection of generalized coordinates $\boldsymbol{\phi}$ and dependent coordinates $\boldsymbol{q}$ always ensures this condition due to the independence of the constraints given in Eq. (1). Note that the first order IKIC matrix is the power conserving transformation (Karnopp 1969) between $\dot{\boldsymbol{\phi}}$ and $\dot{\boldsymbol{q}}$ with its dual form given by

$$\boldsymbol{\tau}_{\phi}=[\tilde{G}]^{T}\boldsymbol{\tau}_{q} \tag{6}$$

where T denotes matrix transpose, $\boldsymbol{\tau}_{\phi}$ and $\boldsymbol{\tau}_{q}$ denote the generalized forces (or efforts) associated with $\boldsymbol{\phi}$ and $\boldsymbol{q}$, respectively.

The second order rate relationship between $\boldsymbol{q}$ and $\boldsymbol{\phi}$ may be obtained by differentiating Eq. (3) w. r. t. time as

$$\ddot{\boldsymbol{q}}=[\tilde{G}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\odot\{\tilde{H}\}\odot\dot{\boldsymbol{\phi}} \tag{7}$$

where

$$\{\tilde{H}\}=\left\{\frac{\partial^{2}\boldsymbol{q}}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}}\right\} \tag{8}$$

$$\begin{aligned}=&-\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\{\frac{\partial^{2}\boldsymbol{f}}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}}\right\}-[\tilde{G}]^{T}\\&\odot\left\{\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\{\frac{\partial^{2}\boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{q}}\right\}\right\}\odot[\tilde{G}]\\&-\left\{\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\{\frac{\partial^{2}\boldsymbol{f}}{\partial\boldsymbol{\phi}\partial\boldsymbol{q}}\right\}\right\}\odot[\tilde{G}]\\&-[\tilde{G}]^{T}\odot\left\{\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\{\frac{\partial^{2}\boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{\phi}}\right\}\right\}\end{aligned} \tag{9}$$

As used in Eqs. (7) and (9), in this paper the delimiter $\{\cdot\}$ denotes a three dimensional tensor and the operators $\odot$ and $\oplus$ were introduced as a generalization of matrix multiplication rule for the expressional compactness and readiness for programming (refer to Appendix for definitions). As an example, a typical three dimensional tensor introduced in Eq. (9) is defined as

$$\left[\frac{\partial^{2}\boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{\phi}}\right]_{kij}=\frac{\partial^{2}f_{k}}{\partial q_{i}\partial\phi_{j}} \tag{10}$$

where indices k, i, and j denote the corresponding plane, row, and column, respectively. The three dimensional tensor $\{\tilde{H}\}\in R^{M\times N\times N}$ will be referred to as the second order IKIC tensor of the system. As it will be shown later, the second order IKIC tensor facilitates evaluation of time rate of change of moduli of various transformers contained in the system.

When the system is constrained by an independent set of nonholonomic constraints, a similar formulation is readily possible as shown in Cho (1994).

2.2 The first order kinematics

Introducing the popular concept of virtual cut, an interconnected multibody system can be made topologically equivalent to a kinematic tree (Huston and Passerello (1979)). Using the kinematic tree, link indices are assigned as follows: First, the index 0 is assigned to an inertially fixed link. Then, the indices of the remaining links can be assigned in such way that they take the same indices as those of joints which initially meet the links along the path defined from link 0 to links of interest. It is important, in this process of assigning link indices, to note that the imaginary massless links arising from modeling compound joints by a series of simple ones should also be assigned proper link indices.

For easy references two additional sets of indices are defined as follows; the set of link indices $V=\{v_i : v_i\in Z \text{ with } i=1, \cdots, v\}$ where Z denotes the set of nonnegative integers and V is the number of links, including imaginary massless links, involved in kinematic tree and the set of body indices $W=\{w_i : w_i\in Z \text{ with } i=1, \cdots, w\}$ where W is the number of physically existing bodies with finite mass contents. Notice that $W\subseteq V\subseteq J\cup\{0\}$.

The basic convention for setting up local coordinate frames at link j is assumed as follows: $\boldsymbol{x}_j$-and $\boldsymbol{y}_j$-axis is defined fixed on link j, collinear with the axis of rotation or direction of sliding depending on whether the joint j is revolute or prismatic, and $\boldsymbol{x}_j$-and $\boldsymbol{y}_j$-axis fixed in the link are properly selected to lie in the plane perpendicular to $\boldsymbol{z}_j$-axis such that usual right-hand rule is satisfied. From now on, all the vectors which appear in equations will be assumed to be expressed w. r. t. the inertial coordinate frame fixed in body 0.

2.2.1 The first order KIC for angular motion

Collecting relative rotational motions between two neighboring links can be collected along appropriate kinematic tree by using the associated connection map, the absolute angular velocity of the link k can be expressed in a matrix form as

$$\omega_k = [\boldsymbol{G}_T^k]\,\dot{\boldsymbol{\phi}} \tag{11}$$

where n^{th} column vector of $[\boldsymbol{G}_T^k] \in R^{3\times N}$, the first order (external) rotational KIC matrices associated with the link k, is given by the weighted linear combination of z_j as

$$[\boldsymbol{G}_T^k]_{:n} = \sum_{j>0}^{k} \alpha_j \{\beta_j \delta^j_{p(n)} + (1-\beta_j)\,[\tilde{\boldsymbol{G}}]_{S(j)n}\}\boldsymbol{z}_j \tag{12}$$

where α_j equals 1 if joint j is revolute or 0 if joint j is prismatic, β_j equals 1 if joint j is independent or 0 if joint j is dependent, and the Kronecker delta $\delta^j_{p(n)}$ equals 1 if $j=p(n)$ or 0 if $j\neq p(n)$, and $8[\tilde{\boldsymbol{G}}]_{\underline{S}(j)n}$ denote $\underline{S}(j)^{th}$ row and n^{th} column element of the first order internal KIC matrix $[\tilde{\boldsymbol{G}}]$, which is equal to $\partial q_{\underline{S}(j)}/\partial\phi_n$ as shown in Eq. (4). Note that the summation in Eq. (12) should be interpreted as being performed downward, starting from link k, along the kinematic tree until the inertially fixed link 0 is reached.

Notice that Eq. (2) defines the power conserving transformation associated with the rotational motion of link k. Its dual form may be given as

$$\boldsymbol{\tau}_\phi = [\boldsymbol{G}_T^k]^T \boldsymbol{\tau}_{w_k} \tag{13}$$

where $\boldsymbol{\tau}_{w_k}$ denotes the generalized force (or effort) vector related with pseudo-coordinate ω_k.

2.2.2 The first order KIC for translational motion

Translational velocity of a point fixed in a body can be expressed in a matrix form. For example, the velocity of the center of mass of k^{th} body is given by

$$\dot{\boldsymbol{P}}_k = [\boldsymbol{G}_t^k]\,\dot{\boldsymbol{\phi}} \tag{14}$$

where the n^{th} column of $[\boldsymbol{G}_t^k] \in R^{3\times N}$, the first order (external) translational KIC matrice associated with the the point of interest, is given by

$$\begin{aligned}[\boldsymbol{G}_t^k]_{:n} = \sum_{j>0}^{k} [\,&(1-\alpha_j)\{\beta_j\delta^j_{p(n)} \\ &+ (1+\beta_j)\,[\tilde{\boldsymbol{G}}]_{S(j)n}\}\boldsymbol{z}_j + \alpha_j\{\beta_j\delta^j_{p(n)} \\ &+ (1-\beta_j)\,[\tilde{\boldsymbol{G}}]_{\underline{S}(j)n}\}\boldsymbol{z}_j \times (\boldsymbol{P}_k - \boldsymbol{R}_j)\,]\end{aligned} \tag{15}$$

where $\boldsymbol{R}_k$ denote the position vector of the origin of the local coordinate frame fixed in link k w. r. t. a reference coordinate frame fixed in link 0. Equation~(14) also defines the power conserving transformation for the translational motion of link k and its dual relation is

$$\boldsymbol{\tau}_\phi = [\boldsymbol{G}_t^k]^T \boldsymbol{\tau}_{P_k} \tag{16}$$

where $\boldsymbol{\tau}_{P_k}$ denotes the generalized force (or effort) vector associated with the position vector $\boldsymbol{P}_k$.

2.3 The second order kinematics

In the bond graph modeling of general interconnected multibody system, the information on the time rate of change of moduli of various modulated transformers are required in the process of constructing state equation of the system. Theoretically, required results may be obtained either by direct symbolic differentiation of moduli of modulated transformers found previously, or by using vector algebra. Here the latter approach is pursued for the sake of systmatic evaluation of them (Cho, 1994).

To find $d/dt([\boldsymbol{G}_T^k])$ in a proper form, notice that

$$\frac{d}{dt}([\boldsymbol{G}_T^k]_{:n}) = \sum_{m=0}^{N} \frac{\partial}{\partial\phi_m}([\boldsymbol{G}_T^k]_{:n})\,\dot{\phi}_{rm} \tag{17}$$

where in view of Eq. (12)

$$\begin{aligned}\frac{\partial}{\partial\phi_m}([\boldsymbol{G}_T^k]_{:n}) = \frac{\partial}{\partial\phi_m}\Big[&\sum_{i>0}^{k}\alpha_i\{\beta_i\delta^i_{p(n)} \\ &+ (1+\beta_i)\,[\tilde{\boldsymbol{G}}]_{\underline{S}(i)n}\}\boldsymbol{z}_i\Big]\end{aligned} \tag{18}$$

$$\begin{aligned}= &\sum_{i>0}^{k}\alpha_i\{\beta_i\delta^i_{p(n)} + (1-\beta_i)\,[\tilde{\boldsymbol{G}}]_{\underline{S}(i)n}\}\frac{\partial\boldsymbol{z}_i}{\partial\phi_m} \\ &+ \sum_{i>0}^{k}\alpha_i(1-\beta_i)\frac{\partial}{\partial\phi_m}([\tilde{\boldsymbol{G}}]_{\underline{S}(i)n})\,\boldsymbol{z}_i\end{aligned} \tag{19}$$

with

$$\frac{\partial\boldsymbol{z}_i}{\partial\phi_m} = \sum_{j>0}^{T(i)}\alpha_j\{\beta_j\delta^j_m + (1-g_j)\,[\tilde{\boldsymbol{G}}]_{\underline{S}(j)m}\}\boldsymbol{z}_j \times \boldsymbol{z}_i, \tag{20}$$

where the mapping $T(\cdot)$, which operates on the set of link indices ν and fully characterizes the interconnection structure of links in a kinematic tree, is called connection map (Huston and Passerello, 1979), and

$$\frac{\partial}{\partial \phi_m}([\tilde{G}_{\underline{S}(i)n}]) = \{\tilde{H}\}_{\underline{S}(i)nm} \tag{21}$$

where $\{\tilde{H}\}_{\underline{S}(i)nm}$ indicates the $\underline{s}(i)^{th}$ plane, n^{th} row, and m^{th} column element of the tensor $\{\tilde{H}\}$ defined in Eq. (8). Now, the n^{th} column vector of $d/dt([G_T^k])$ may be expressed as

$$\left[\frac{d}{dt}([G_T^k])\right]_n = [\{H_T^k\}_{n:}]\dot{\phi} \tag{22}$$

where $[\{H_T^k\}_{n:}] \in R^{3 \geq N}$ is the n^{th} plane of the tensor $\{H_T^k\} \in R^{N \times 3 \times N}$ called the second order rotational KIC tensor. The m^{th} column vector of n^{th} plane of the tensor $\{H_T^k\}$, denoted by $\{H_T^k\}_{n:m}$, is constructed by the linear combination of vectors as

$$\{H_T^k\}_{n:m} = \frac{\partial}{\partial \phi_m}([G_T^k]_n) \tag{23}$$

$$= \sum_{i>0}^{k} \sum_{j>0}^{T(i)} \alpha_i \alpha_j \{g_i \delta^i_{p(n)} + (1-\beta_i)[\tilde{G}]_{\underline{S}(i)n}\} \{\beta_j \delta^j_{p(m)} + (1-\beta_j)[\tilde{G}]_{\underline{S}(j)m}\} z_j \times z_i + \sum_{i>0}^{k} \alpha_i (1-\beta_i)\{\tilde{H}\}_{\underline{S}(i)nm} z_i. \tag{24}$$

Finally, $d/dt([G_T^k]) \in R^{3 \times N}$ may be expressed in column-wise form as

$$[\dot{G}_T^k] = \left[[\{H_T^k\}_{1:}]\dot{\phi} \vdots [\{H_T^k\}_{2:}]\dot{\phi} \vdots \cdots \vdots [\{H_T^k\}_{N:}]\dot{\phi} \right] \tag{25}$$

It can be proceeded similarly to find $d/dt([G_t^k])$ in the following form.

$$\left[\frac{d}{dt}([G_t^k])\right]_n = [\{H_t^k\}_{n:}]\dot{\phi} \tag{26}$$

where the second order translational KIC tensor $\{H_t^k\} \in R^{N \times 3 \times N}$ is formed by collecting vectors as shown in Cho (1994). Hence, $d/dt([G_t^k]) \in R^{3 \times N}$ may be expressed in column-wise form as

$$[\dot{G}_t^k] = \left[[\{H_t^k\}_{1:}]\dot{\phi} \vdots [\{H_t^k\}_{2:}]\dot{\phi} \vdots \cdots \vdots [\{H_t^k\}_{N:}]\dot{\phi} \right]. \tag{27}$$

3. Hamiltonian Bond Graph

3.1 Basic structure with pure interial elements

The basic structure for Hamiltonian bond graph may be constructed from pure inertial interconnected multibody system. Resulting Hamiltonian bond graph structure serves as the backbone of the bond graph approach developed here.

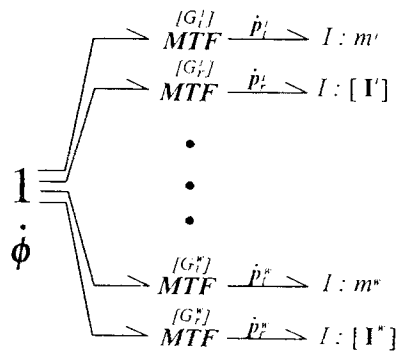

Fig. 2 Bond graph of interconnected inertial system.

Figure 2 shows the conventional bond graph of interconnected multibody system, where only the inertial elements are included and other elements, e. g., compliance, resistor, and gravity, etc., will be considered later. In the Fig. $\boldsymbol{p}_t^k$ and $\boldsymbol{p}_r^k$ for $k \in W$ denote translational (or linear) and rotational (or angular) momentum vectors, respectively, of k^{th} body with finite mass contents, m^k, $[\boldsymbol{I}^k]$ stands for mass moment of inertia of body i w. r. t. the inertial coordiante frame, and the superscript W denotes the total number of bodies with finite mass contents. Note that modulated transformers (MTF) are modulated by the signal vector of Lagrangian coordinate Ψ, i. e., the generalized coordinate $\boldsymbol{\phi}$ and dependent coordinate $\boldsymbol{q}$.

Although not shown explicitly in Fig. 2, it is well known that generally not all inertial elements are independent in the sense of causality. If, by starting from an inertial element, the integral causalities are assigned in the conventional sense and propagated, the dependent inertial elements appear by coupling through the 1-junction with common flow $\dot{\boldsymbol{\phi}}$. In this process of assigning causalities, the moduli of modulated transformers must be taken into consideration. This is because at given configuration of the system, some of the moduli determined by the signal vectors $\boldsymbol{\phi}$ and $\boldsymbol{q}$ may cause difficulty in expressing the states of dependent inertial elements in terms of indepen-

dent ones.

A novel way of getting around this difficulty is to formulate in terms of the equivalent (or effective) inertial field seen from 1-junction of $\dot{\boldsymbol{\phi}}$. This approach, which naturally leads to the Hamiltonian bond graph, is attempted here. $\boldsymbol{e}_I \in R^N$ be the effort vector coming into the 1-junction of $\dot{\boldsymbol{\phi}}$ from inertial elements through MTF's. Then, by noting in Fig. 2 that $\dot{\boldsymbol{p}}_t^k$ and $\dot{\boldsymbol{p}}_r^k$ for $k \in W$ are the efforts signal related with inertial elements, it is easy to see that

$$\boldsymbol{e}_I = \sum_{k \in W} ([\boldsymbol{G}_t^k]^T \dot{\boldsymbol{p}}_t^k + [\boldsymbol{G}_r^k]^T \dot{\boldsymbol{p}}_r^k) \tag{28}$$

where the summation is performed over the set of body indices W. Note that $[\boldsymbol{G}_t^k]$ and $[\boldsymbol{G}_t^{k^*}]$ are found previously as in Eq. (12) and (15), respectively. To find the equivalent inertial field, rewrite Eq. (28))

$$\begin{aligned} \boldsymbol{e}_I = & \frac{d}{dt}\left\{\sum_{k \in W} ([\boldsymbol{G}_t^k]^T \boldsymbol{p}_t^k + [\boldsymbol{G}_r^k]^T \boldsymbol{p}_r^k)\right\} \\ & - \sum_{k \in W} \{[\dot{\boldsymbol{G}}_t^k]^T \boldsymbol{p}_t^k + [\dot{\boldsymbol{G}}_r^k]^T \boldsymbol{p}_r^k\} \end{aligned} \tag{29}$$

where $[\dot{\boldsymbol{G}}_r^k]$ and $[\dot{\boldsymbol{G}}_t^k]$ are given by Eq. (25) and (27), respectively. Let's define the generalized momentum vector $\boldsymbol{p} \in R^N$ of the system as

$$\boldsymbol{p} = \sum_{k \in W} ([\boldsymbol{G}_t^k]^T \boldsymbol{p}_t^k + [\boldsymbol{G}_r^k]^T \boldsymbol{p}_r^k). \tag{30}$$

Now, noting that

$$\boldsymbol{p}_t^k = m^k [\boldsymbol{G}_t^k] \dot{\boldsymbol{\phi}} \tag{31}$$

and

$$\boldsymbol{p}_r^k = [\boldsymbol{I}^k][\boldsymbol{G}_r^k] \dot{\boldsymbol{\phi}}, \tag{32}$$

the generalized momentum vector $\boldsymbol{p}$ may be rewritten as

$$\boldsymbol{p} = \left[\sum_{k \in W} (m^k [\boldsymbol{G}_t^{k^*}]^T [\boldsymbol{G}_t^{k^*}] + [\boldsymbol{G}_r^k]^T [\boldsymbol{I}^k][\boldsymbol{G}_r^{k^*}])\right] \dot{\boldsymbol{\phi}} \tag{33}$$

$$= [\boldsymbol{I}^*] \dot{\boldsymbol{\phi}} \tag{34}$$

where $[\boldsymbol{I}] \in R^{N \times N}$ is the inertial field matrix of the system defined by

$$\begin{aligned} [\boldsymbol{I}^*] = & \sum_{k \in W} (m^k [\boldsymbol{G}_t^k]^T [\boldsymbol{G}_t^k] + [\boldsymbol{G}_r^k]^T \cdot \\ & [\boldsymbol{I}^k][\boldsymbol{G}_r^k]) \end{aligned} \tag{35}$$

In terms of $[\boldsymbol{I}^*]$, the kinetic energy KE and kinetic coenergy KE^* of an inertial system can be expressed as

$$KE = \frac{1}{2} \boldsymbol{p}^T [\boldsymbol{I}^*]^{-1} \boldsymbol{p} \tag{36}$$

and

$$KE^* = \frac{1}{2} \dot{\boldsymbol{\phi}}^T [\boldsymbol{I}^*] \dot{\boldsymbol{\phi}} \tag{37}$$

Note that the kinetic energy KE, which is the integral of flow with respect to generalized momentum, and the kinetic coenergy KE^* defined by the integral of generalized momentum with respect to flow are related by Legendre transformation.

Substituting Eqs. (31) and (32) into Eq. (29) and using Eq. (34) lead successively to the following relations.

$$\begin{aligned} \boldsymbol{e}_I = & \dot{\boldsymbol{p}} - \left[\sum_{k \in W} \{m^k [\dot{\boldsymbol{G}}_t^k]^T [\boldsymbol{G}_t^k] + [\dot{\boldsymbol{G}}_t^k]^T \right. \\ & \left. [\boldsymbol{I}^k][\boldsymbol{G}_t^k]\}\right] \dot{\boldsymbol{\phi}} \end{aligned} \tag{38}$$

$$\begin{aligned} = & \dot{\boldsymbol{p}} - \left[\sum_{k \in W} \{m^k [\dot{\boldsymbol{G}}_t^k]^T [\boldsymbol{G}_t^k] + [\dot{\boldsymbol{G}}_t^k]^T \cdot \right. \\ & \left. [\boldsymbol{I}^k][\boldsymbol{G}_r^k]\}\right] [\boldsymbol{I}^*]^{-1} \boldsymbol{p} \end{aligned} \tag{39}$$

or

$$\dot{\boldsymbol{p}} = \boldsymbol{e}_I + [\boldsymbol{GR}][\boldsymbol{I}^*]^{-1} \boldsymbol{p} \tag{40}$$

where the matrix $[\boldsymbol{GR}] \in R^{N \times N}$ forms the gyristor field (Allen 1979) and is defined as

$$\begin{aligned} [\boldsymbol{GR}] = & \left[\sum_{k \in W} \{m^k [\dot{\boldsymbol{G}}_t^{k^*}]^T [\boldsymbol{G}_t^{k^*}] + [\dot{\boldsymbol{G}}_r^{k^*}]^T \cdot \right. \\ & \left. [\boldsymbol{I}^k][\boldsymbol{G}_r^{k^*}]\}\right] \end{aligned} \tag{41}$$

Notice that elements of the gyristor matrix $[\boldsymbol{GR}]$ are the function of $\boldsymbol{\phi}$, $\boldsymbol{q}$, and $\dot{\boldsymbol{\phi}}$ (or the generalized momentum $\boldsymbol{p}$).

Notice that Eq. (40) is, in fact, the Hamilton's equation derived using bond graph. In view of Eq. (40) it may be observed that the total effect of the inertial elements of an interconnected multibody system can be effectively modeled by using the equivalent inertial field matrix $[\boldsymbol{I}^*]$ defined w. r. t. the generalized flow vector $\dot{\boldsymbol{\phi}}$ and a gyristor field matrix $[\boldsymbol{GR}]$ as shown in the Fig. 3. Based on the development so far a general interconnected multibody system can be represented in the Hamiltonian bond graph as in Fig. 4 with natural causality assigned. In Fig. 4, Q represents the generalized effort vector caused

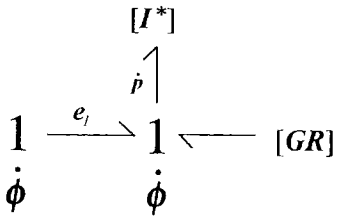

Fig. 3 Equivalent inertial field model.

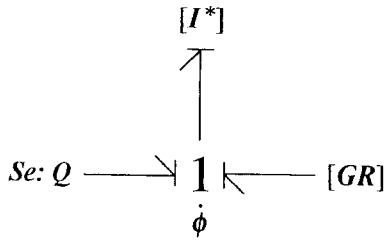

Fig. 4 Bond graph Representation of general multibody system.

by both conservative and nonconservative force field as will be shown successively. Hence, the state equation of the system can be written as

$$\dot{\boldsymbol{p}}=-[\boldsymbol{GR}(\boldsymbol{\phi},\boldsymbol{q},\boldsymbol{p})][\boldsymbol{I}^*(\boldsymbol{\phi},\boldsymbol{q})]^{-1}\boldsymbol{p}+\boldsymbol{Q} \quad (42)$$

$$\dot{\boldsymbol{\phi}}=[\boldsymbol{I}^*(\boldsymbol{\phi},\boldsymbol{q})]^{-1}\boldsymbol{p} \quad (43)$$

$$\dot{\boldsymbol{q}}=[\tilde{\boldsymbol{G}}(\boldsymbol{\phi},\boldsymbol{q})][\boldsymbol{I}^*(\boldsymbol{\phi},\boldsymbol{q})]^{-1}\boldsymbol{p}. \quad (44)$$

Notice that the Eq. (44) is obtained from Eq. (3) where $[\tilde{\boldsymbol{G}}(\boldsymbol{\phi},\boldsymbol{q})]$ is given by Eq. (5) for holonomic case and should be included in the state equation in order to update the dependent coordinates $\boldsymbol{q}$ in time.

To show that the same result can be obtained directly from Hamilton's equation, define the Lagrangian L and, by Legendre's transformation, the Hamiltonian H of the system as

$$L=\frac{1}{2}\dot{\boldsymbol{\phi}}^T[\boldsymbol{I}^*]\dot{\boldsymbol{\phi}} \quad (45)$$

$$H=\frac{1}{2}\dot{\boldsymbol{p}}^T[I^*]^{-1}\dot{\boldsymbol{p}} \quad (46)$$

Then, the Hamilton's equation gives

$$\dot{\boldsymbol{p}}=-\frac{\partial H}{\partial \boldsymbol{\phi}}+Q \quad (47)$$

where $Q\in R^N$ denotes the generalized effort vector including conservative forces and

$$\frac{\partial H}{\partial \boldsymbol{\phi}}=-\frac{\partial L}{\partial \boldsymbol{\phi}} \quad (48)$$

$$=-\frac{\partial}{\partial \boldsymbol{\phi}}\left(\frac{1}{2}\dot{\boldsymbol{\phi}}^T[\boldsymbol{I}^*]\dot{\boldsymbol{\phi}}\right) \quad (49)$$

In evaluating Eq. (49), many different schemes may be employed. One scheme defined below in Eq. (54), probably the simplest for the hand evaluation, yields

$$\frac{\partial}{\partial \boldsymbol{\phi}}\left(\frac{1}{2}\dot{\boldsymbol{\phi}}^T[\boldsymbol{I}]\dot{\boldsymbol{\phi}}\right)=\frac{1}{2}\dot{\boldsymbol{\phi}}^T\odot\{\boldsymbol{P}^*\}\odot\dot{\boldsymbol{\phi}} \quad (50)$$

$$=[\tilde{\boldsymbol{GR}}]\dot{\boldsymbol{\phi}} \quad (51)$$

$$=[\tilde{\boldsymbol{GR}}][\boldsymbol{I}^*]^{-1}\boldsymbol{p} \quad (52)$$

where another form of gyristor field matrix $[\tilde{\boldsymbol{GR}}]\in R^{N\times N}$ is produced in row vector form as

$$[\tilde{\boldsymbol{GR}}]=\begin{bmatrix}\dot{\boldsymbol{\phi}}^T[\{\boldsymbol{P}^*\}_{1::}]\\ \cdots\\ \dot{\boldsymbol{\phi}}^T[\{\boldsymbol{P}^*\}_{2::}]\\ \cdots\\ \vdots\\ \cdots\\ \dot{\boldsymbol{\phi}}^T[\{\boldsymbol{P}^*\}_{N::}]\end{bmatrix} \quad (53)$$

with the three dimensional tensor $[\boldsymbol{P}^*]\in R^{N\times N\times N}$ being defined as

$$\{\boldsymbol{P}^*\}_{kij}=\frac{\partial}{\partial \phi_k}([\boldsymbol{I}^*]_{ij}). \quad (54)$$

It should be pointed out that, although as mentioned previously other forms of gyristor field matrix are possible depending on the method of evaluation of Eq. (49)), the net dynamic effects of the resulting gyristor are always the same. In other words, they produce the same effort vector in response to the input flow vector $\dot{\boldsymbol{\phi}}$. In fact, it can be proved that although the gyristor obtained through direct bond graph manipulation as in Eq. (41) and the gyristor in Eq. (53) obtained from direct application of Hamilton's equation are generally different in their elements, the efforts they produce in response to flow $\dot{\boldsymbol{\phi}}$ are the same, i. e.,

$$[\boldsymbol{GR}]\dot{\boldsymbol{\phi}}=[\tilde{\boldsymbol{GR}}]\dot{\boldsymbol{\phi}} \quad (55)$$

3.2 Modeling of compliance element

In Fig. 5 typical translational and rotational compliance elements are shown, in which K_i denotes the translational or rotational stiffness of the i^{th} linear spring, $\boldsymbol{P}_C^{1i}$ and $\boldsymbol{P}_C^{2i}$ indicate position vectors of two ends of the spring represented w. r.

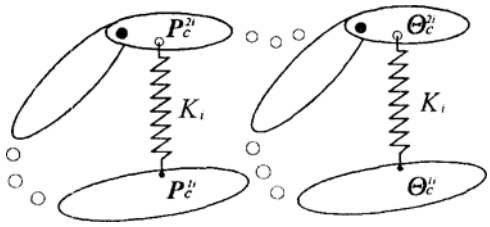

Fig. 5 Typical compliance element.

t. inertially fixed coordinate, and $\boldsymbol{\Theta}_C^{1i}$ and $\boldsymbol{\Theta}_C^{2i}$ are assumed to represent angular displacements of two links to which i^{th} rotational spring is attached. Let $\boldsymbol{q}_C^i \in R^3$ be the relative displacement vector of the i^{th} spring defined by

$$\boldsymbol{q}_C^i = \begin{cases} \boldsymbol{P}_C^{2i} - \boldsymbol{P}_C^{1i} : \text{translational} \\ \boldsymbol{\Theta}_C^{2i} - \boldsymbol{\Theta}_C^{1i} : \text{rotational} \end{cases} \tag{56}$$

$$= q_C^i \boldsymbol{n}_C^i \tag{57}$$

where q_C^i is the magnitude of $\boldsymbol{q}_C^i$, i. e., $\|\boldsymbol{q}_C^i\|$ and the unit vector $\boldsymbol{n}_C^i \in R^3$ is defined by

$$\boldsymbol{n}_C^i = \boldsymbol{q}_C^i / q_C^i. \tag{58}$$

Let the constant q_C^{i0} be the free translational or rotational displacement of the i^{th} spring. Then, the generalized effort vector from compliance elements, coming into the 1-junction of $\dot{\phi}$, denoted by $\boldsymbol{e}_C \in R^N$, may be found as

$$\boldsymbol{e}_C = -\sum_{i=1}^{K} K_i [\hat{\boldsymbol{G}}_C^i]^T \Delta \boldsymbol{q}_C^i \tag{59}$$

where the net displacement vector $\Delta \boldsymbol{q}_C^i \in R^3$ of the i^{th} spring is defined by

$$\Delta \boldsymbol{q}_C^i = (q_C^i - q_C^{i0}) \boldsymbol{n}_C^i \tag{60}$$

and the matrix $[\hat{\boldsymbol{G}}_C^i] \in R^{3\times N}$ of the required transformer, whose elements are in general the functions of $\boldsymbol{\phi}$ and $\boldsymbol{q}$, may be found by

$$[\hat{\boldsymbol{G}}_C^i] = \frac{\partial \boldsymbol{q}_C^i}{\partial \boldsymbol{\phi}} \tag{61}$$

$$= \begin{cases} \partial(\boldsymbol{P}_C^{2i} - \boldsymbol{P}_C^{1i})/\partial\boldsymbol{\phi} = \partial(\dot{\boldsymbol{P}}_C^{2i} - \dot{\boldsymbol{P}}_C^{1i})/\partial\dot{\boldsymbol{\phi}} : \text{translational} \\ \partial(\boldsymbol{\Theta}_C^{2i} - \boldsymbol{\Theta}_C^{1i})/\partial\boldsymbol{\phi} = \partial(\boldsymbol{\omega}_C^{2i} - \boldsymbol{\omega}_C^{1i})/\partial\dot{\boldsymbol{\phi}} : \text{rotational.} \end{cases} \tag{62}$$

Notice that $\partial \dot{\boldsymbol{P}}_C^{ji}/\partial\dot{\boldsymbol{\phi}}$ for $j=1, 2$, are the translational KIC of the end points of the i^{th} spring, which can be expressed in the same form as in Eq. (15) and $\partial \boldsymbol{\omega}_C^{ji}/\partial\dot{\boldsymbol{\phi}}$ for $j=1, 2$ are the rotational KIC, also expressible in the form of Eq. (12), of the links to which each end point of i^{th} rotational spring is attached. Now noting also that $[\hat{\boldsymbol{G}}_C^i]$ may also be expressed in column-wise form as

$$[\hat{\boldsymbol{G}}_C^i] = \left[\frac{\partial q_C^i}{\partial \phi_1} \boldsymbol{n}_C^i + q_C^i \frac{\partial \boldsymbol{n}_C^i}{\partial \phi_1} \vdots \frac{\partial q_C^i}{\partial \phi_2} \boldsymbol{n}_C^i + q_C^i \frac{\partial \boldsymbol{n}_C^i}{\partial \phi_2} \vdots \cdots \vdots \frac{\partial q_C^i}{\partial \phi_N} \boldsymbol{n}_C^i + q_C^i \frac{\partial \boldsymbol{n}_C^i}{\partial \phi_N} \right], \tag{63}$$

and the fact that the scalar product of two vectors, $\partial \boldsymbol{n}_C^i/\partial\phi_j$ and $\boldsymbol{n}_C^i$, vanishes for $j=1, \cdots, N$, the expression in Eq. (59) may be rewritten as

$$\boldsymbol{e}_C = -\sum_{i=1}^{K} K_i [\boldsymbol{G}_C^i]^T \Delta \boldsymbol{q}_C^i \tag{64}$$

where the new modulus matrix $[\boldsymbol{G}_C^i] \in R^{3\times N}$ is defined as

$$[\boldsymbol{G}_C^i] = \left[\frac{\partial q_C^i}{\partial \phi_1} \boldsymbol{n}_C^i \vdots \frac{\partial q_C^i}{\partial \phi_2} \boldsymbol{n}_C^i \vdots \cdots \vdots \frac{\partial q_C^i}{\partial \phi^N} \boldsymbol{n}_C^i \right] \tag{65}$$

$$= \frac{\partial \Delta \boldsymbol{q}_C^i}{\partial \boldsymbol{\phi}} \tag{66}$$

by eliminating $\partial \boldsymbol{n}_C^i/\partial\phi_i$ component in columns of the matrix $[\hat{\boldsymbol{G}}_C^i]$.

Notice that although Eqs. (59) and (64) provides the same information about the effort vector due to compliance elements in the system and can be substituted into Q in Eq. (42). In fact every time varying quantity in Eqs. (59) and (64) may generally be expressed as a function of $\boldsymbol{\phi}$ and $\boldsymbol{q}$, which are the states of the system as in Eqs. (43) and (44). For example, Eq. (64) may be written with explicit functional dependencies indicated as

$$\boldsymbol{e}_C = -\sum_{i=1}^{K} K_i [\boldsymbol{G}_C^i(\boldsymbol{\phi}, \boldsymbol{q})]^T \Delta \boldsymbol{q}_C^i(\boldsymbol{\phi}, \boldsymbol{q}). \tag{67}$$

Another view of linear compliance elements with nonlinear geometry may be given by observing their local behavior around a given system configuration defined by $\boldsymbol{\phi}$ and $\boldsymbol{q}$. Let $[\boldsymbol{K}(\boldsymbol{\phi}, \boldsymbol{q})] \in R^{N\times N}$ be the local stiffness reflected at 1-junction of $\dot{\boldsymbol{\phi}}$ at given $\boldsymbol{\phi}$ and $\boldsymbol{q}$. Then,

$$[\boldsymbol{K}] = -\frac{\partial \boldsymbol{e}_C}{\partial \boldsymbol{\phi}} \tag{68}$$

$$= \left[\sum_{i=1}^{K} [\boldsymbol{G}_C^i]^T K_i \frac{\partial \Delta \boldsymbol{q}_C^i}{\partial \boldsymbol{\phi}} \right] + \left[\sum_{i=1}^{K} K_i (\Delta \boldsymbol{q}_C^i)^T \odot \left\{ \frac{\partial [\boldsymbol{G}_C^i]}{\partial \boldsymbol{\phi}} \right\} \right] \tag{69}$$

$$= \left[\sum_{i=1}^{K} K_i [\boldsymbol{G}_C^i]^T [\boldsymbol{G}_C^i] \right] + \left[\sum_{i=1}^{K} K_i (\Delta \boldsymbol{q}_C^i)^T \right.$$

$$\odot\left\{\frac{\partial[\boldsymbol{G}_C^i]}{\partial\boldsymbol{\phi}}\right\}\Big] \tag{70}$$

where the three dimensional tensor $\{\partial[\boldsymbol{G}_C^i]/\partial\boldsymbol{\phi}\}\in R^{N\times 3\times N}$ is defined by

$$\left\{\frac{\partial[\boldsymbol{G}_C^i]}{\partial\boldsymbol{\phi}}\right\}_{lmn}=\frac{\partial}{\partial\phi_n}([\boldsymbol{G}_C^i]_{lm}) \tag{71}$$

and can also be expressed by using the second order KIC tensors. Equation (70) shows that $[\boldsymbol{K}]$ consists of two matrices. The first stiffness matrix is usually called the equivalent stiffness matrix reflected through the transformer with modulus $[\boldsymbol{G}_C^i]$, and the second is caused by the spring forces at given configuration, i. e., preloading forces, and nonlinear geometry involved in transformers. The latter stiffness effect can be called an antagonistic stiffness because it may exist in any redundantly actuated system, i. e., the system with larger number of independent effort sources than its kinematic degrees of freedom.

5.3 Modeling of resistive element

Linear translational and rotational resistance elements are shown in Fig. 6, in which R_i denotes translational or rotational damping coefficient. Let $\boldsymbol{e}_R\in R^N$ be the effort vector coming into 1-junction of $\dot{\boldsymbol{\phi}}$ Then,

$$\boldsymbol{e}_R=\sum_{i=1}^{n}[\boldsymbol{G}_R^i]^T R_i[\boldsymbol{G}_R^i]\,\dot{\boldsymbol{\phi}} \tag{72}$$

where R is the total number of resistance elements in the system and $[\boldsymbol{G}_R^i]$ is the modulus matrix of the transformer between the i^{th} resistance element and the 1-junction of $\dot{\boldsymbol{\phi}}$, which can be easily determined by using the first order KIC matrices similar to the case of compliance elements. In terms of the equivalent resistance field matrix, Eq. (72) may be rewritten as

$$\boldsymbol{e}_R=[\boldsymbol{R}]\,\dot{\boldsymbol{\phi}} \tag{73}$$

where the equivalent resistance field matrix $[\boldsymbol{R}(\boldsymbol{\phi},\boldsymbol{q})]\in R^{N\times N}$ is defined as

$$[\boldsymbol{R}(\boldsymbol{\phi},\boldsymbol{q})]=\sum_{i=1}^{R}R_i[\boldsymbol{G}_R^i]^T[\boldsymbol{G}_R^i]. \tag{74}$$

5.4 Modeling of gravitational effort source elements

To complete the modeling of an interconnected multibody system, the gravitational forces must be included. Let the effort vector reflected to the 1-junction of $\dot{\boldsymbol{\phi}}$ due to gravity be $\boldsymbol{e}_G$. Then, $\boldsymbol{e}_G$ may be expressed as

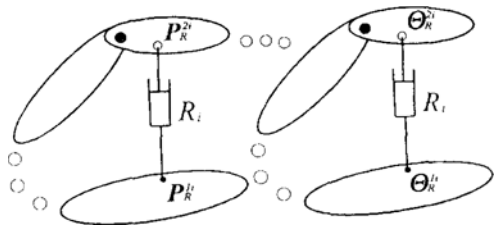

Fig. 6 Typical resistive element.

$$\boldsymbol{e}_G=\sum_{i=1}^{n}m_i[\boldsymbol{G}_I^i]^T\boldsymbol{g}_i \tag{75}$$

where $[\boldsymbol{G}_I^i]\in R^{3\times N}$ is the modulus matrix of the i^{th} inertial element, which can be found by exactly the same way as in Eq. (15), and $\boldsymbol{g}_i\in R^3$ is the gravitational acceleration vector of the i^{th} inertial element expressed w. r. t. the inertial coordinate frame.

6. Conclusion

In this paper, a systematic procedure for modeling of general interconnected multibody dynamic system was developed in bond graph environments. The final state equation is given in Eqs. (42) through (44). In Eq. (42), the total generalized effort vector Q is determined as

$$Q=Q^*+\boldsymbol{e}_C+\boldsymbol{e}_R+\boldsymbol{e}_G \tag{76}$$

where the efforts vectors $\boldsymbol{e}_C$, $\boldsymbol{e}_R$ and $\boldsymbol{e}_G$ are defined respectively in Eq. (59) (or (64), (72), and (75), and the vector $Q^*\in R^N$ denotes the equivalent effort reflected from other effort sources which might exist in the system.

Several features of the proposed approach are listed as follows: First, the formulation started from the bond graph structure proposed by Rosenberg (1972), which seems to be the standard as in Karnopp et al. (1990). This structure has a distinct advantage of systematically visualizing interconnection structure of inertial elements through generalized coordinates in a compact format. Second, although the structure of the final bond graph can be given by an extremely simple

Hamiltonian form similar to the one given in Allen (1979) and Bos and Tiernego (1985), the details of the energetic structure of the system can be shown whenever necessary. Third, all modulated transformers and their time rate of changes are found by pure vector (matrix) operations. This implies the readiness for computer simulation of the resulting bond graph.

Acknowledgement

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Appendix

Algebraic definitions of the operators $\oplus$ and $\odot$ are provided as follows: Given a matrix $[A]\in R^{n\times k}$ and tensor $\{B\}\in R^{k\times m\times m}$, new tensor $\{C\}\in R^{n\times m\times m}$ is formed by $\oplus$ operations as

$$\{C\}_{ijk}=[A]_{ij}\oplus\{B\}_{jkl} \tag{77}$$

$$=\sum_{j=1}^{k}[A]_{ij}\{B\}_{jkl} \tag{78}$$

Basically, $\oplus$ operation makes each plane of tensor $[B]$ uniformly scaled by the low element of $[A]$ and then summed. Note that $\{B\}\oplus[A]_{ij}$ is not defined.

The operation $\odot$ is the extension of usual matrix multiplication rule to three dimensional tensor. Typical examples are as follows. A three dimensional quadratic operation is defined for a vector $b\in R^m$ and the three dimensional tensor $\{B\}\in R^{k\times m\times m}$ to yield another vector $a\in R^k$ as

$$a=b^T\odot\{B\}\odot b \tag{79}$$

$$=\begin{pmatrix} b^T\{B\}_{1jk}\,b \\ b^T\{B\}_{2jk}\,b \\ \vdots \\ b^T\{B\}_{kjk}\,b \end{pmatrix}. \tag{80}$$

Multiplication of a matrix $[A]\in R^{n\times m}$ and a tensor $\{B\}\in R^{k\times m\times l}$ yields a tensor $\{C\}\in R^{k\times n\times l}$ as

$$\{C\}_{ijk}=[A]\odot\{B\} \tag{81}$$

$$=[A]\{B\}_{ijk} \tag{82}$$

Similarly $\{B\}\odot\{A\}$ is defined with $[A]\in R^{m\times l}$ and $\{B\}\in R^{k\times n\times m}$ as

$$\{C\}_{ijk}=\{B\}\odot[A] \tag{83}$$

$$=\{B\}_{ijk}[A] \tag{84}$$

where $\{C\}$ becomes $k\times n\times l$ tensor. No $\odot$ operation is defined on any pair of three dimensional tensors.

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